Relating Kleene algebras with pseudo uninorms

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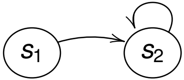
Outline

Introduction: Kleene Algebra

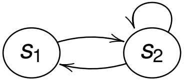
- Kleene Algebra (KA) are considered the standard abstraction of a computational system.
- For instance, by starting from the atomic programs represented in the transition systems of Fig. 1

A_π :



$$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$A_{\pi'}$:



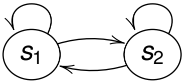
$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

Figure: Examples of abstract programs

Introduction: Kleene Algebra (continuation)

- The nondeterministic choice $+$ and iteration closure $*$ operations of a KA are needed to encode imperative programs.
- the sequential composition ; of a KA can interpret the abstract program $A_{\pi;\pi'}$ in Fig. 2 which compute one step in A_π followed by another in $A_{\pi'}$.

$A_{\pi;\pi'}$:



$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Figure: Example of composition of abstract programs

Introduction: Pseudo Uninorms

- Aggregation functions (AF) over a bounded lattice are increascing functions which preserve the bounds of the lattice.
- The purpose of AF's is to combine inputs that are typically interpreted as membership degrees in L-fuzzy sets.
- The AF can be classified in conjunctive, disjunctive, average and mixed.
- Proper Uninorms is the most important class of mixed AF's, and can be constructed from a t-norm (conjunctive AF) and t-conorm (disjunctive AF).
- t-norms, t-conorm and uninorms had been extensively studied.
- All of them are symmetric functions, but in some problems the symmetry is unnecessary.
- The pseudo t-norms, pseudo t-conorms and pseudo uninorm arose removing the symmetry property of such types of AF's.

Introduction: Contributions of this work

- This paper develops a novel algebraic study on Kleene algebras, based on pseudo uninorms defined over partial orders.
- We investigate how fuzzy programs, i.e., elements of these structures, behaves with respect to the Kleene operations. At this level, classic choice is maintained, but the notions of sequential composition and Kleene closure are abstracted as specific uninorms, defined over meet semilattices.
- We introduce an operator to construct new Kleene algebras from other Kleene algebras based in the notion of automorphism.

Preliminaries: Notations and basic definitions

- In all this talk $\langle P, \leq \rangle$ will be a poset, $e \in P$, $P_e = \{x \in P : x \leq e\}$, $P^e = \{x \in P : e \leq x\}$, $\leq_e$ and $\leq^e$ the restriction of $\leq$ to P_e and P^e, $\Delta_P = \{a \in P : \text{for each } x \in P \; a \leq x \text{ or } x \leq a\}$.

Definition

A **closure operator** on P is a function $\star : P \to P$ such that for each $x, y \in P$

(C1) if $x \leq y$ then $x^\star \leq y^\star$ — Isotonicity,

(C2) $x \leq x^\star$ — inflation, and

(C3) $(x^\star)^\star = x^\star$ — idempotency.

Preliminaries: Pseudo Uninorms

Definition

Let $\langle P, \leq \rangle$ be a poset. A function $U : P \times P \to P$ is a **pseudo uninorm** on P, whenever, for each $w, x, y, z \in P$ it satisfies:

(PU1) $U(x, U(y, z)) = U(U(x, y), z)$ — **Associativity**,

(PU2) $w \leq x$ and $y \leq z$ then $U(w, y) \leq U(x, z)$ — **Isotonicity**, and

(PU3) there is $e \in P$ s.t. $U(x, e) = U(e, x) = x$ — **has neutral element**.

- $\mathfrak{U}_P^e$ is the set of all pseudo uninorms on P with neutral element e.
- If e is the greater (least) element of P then U is called of **pseudo t-norm** (**pseudo t-conorm**).
- Commutative pseudo uninorms are called of uninorms.

New Results on Pseudo Uninorms

Proposition

Let $\langle P, \leq, \bot, \top \rangle$ be a bounded poset and $e \in P$. If $\langle P^e, \leq^e \rangle$ is a join-semilattice and $\langle P_e, \leq_e \rangle$ is a meet-semilattice, then $\langle \mathfrak{U}_P^e, \preceq_e \rangle$ bounded poset.

Proof.

$$\underline{U}_e(x,y) = \begin{cases} \bot & \text{if } x, y \notin P^e \\ x \vee y & \text{if } x, y \in P^e \\ x & \text{if } x \notin P^e \text{ and } y \in P^e \\ y & \text{if } x \in P^e \text{ and } y \notin P^e \end{cases}$$

$$\overline{U}_e(x,y) = \begin{cases} x \wedge y & \text{if } x, y \in P_e \\ \top & \text{if } x, y \notin P_e \\ x & \text{if } x \notin P_e \text{ and } y \in P_e \\ y & \text{if } x \in P_e \text{ and } y \notin P_e \end{cases}$$

□

New Results on Pseudo Uninorms

Proposition

Let $e \in P$ and $U \in \mathfrak{U}_P^e$. Then the restriction, $U_{/P_e}$, is a pseudo t-norm on $\langle P_e, \leq_e \rangle$ and $U_{/P^e}$ is a pseudo t-conorm on $\langle P^e, \leq^e \rangle$.

Corollary

Let $e \in P$, $U \in \mathfrak{U}_P^e$. Then for each isotone bijection $\phi : P_e \to P$, the function $T : P \times P \to P$ defined by

$$T(x, y) = \phi(U(\phi^{-1}(x), \phi^{-1}(y)))$$

is a pseudo t-norm on P. Analogously, for each isotone bijection $\psi : P^e \to P$, the function $S : P \times P \to P$ defined by

$$S(x, y) = \psi(U(\psi^{-1}(x), \psi^{-1}(y)))$$

is a pseudo t-conorm on P.

New Results on Pseudo Uninorms

Proposition

Let $\langle P, \leq \rangle$ be a poset, $e \in P$ and $U_1, U_2 \in \mathfrak{U}_P^e$. Then the mappings $U_1 \rtimes U_2, U_1 \ltimes U_2 : P \times P \to P$ defined by

$$U_1 \rtimes U_2(x,y) = \begin{cases} U_1(x,y) & \text{if } x, y \in P_e \\ U_2(x \vee e, y \vee e) & \text{if } x, y \notin P_e \\ x & \text{if } x \notin P_e \text{ and } y \in P_e \\ y & \text{if } x \in P_e \text{ and } y \notin P_e \end{cases} \quad (1)$$

$$U_1 \ltimes U_2(x,y) = \begin{cases} U_1(x \wedge e, y \wedge e) & \text{if } x, y \notin P^e \\ U_2(x,y) & \text{if } x, y \in P^e \\ x & \text{if } x \notin P^e \text{ and } y \in P^e \\ y & \text{if } x \in P^e \text{ and } y \notin P^e \end{cases} \quad (2)$$

are pseudo uninorms on P with e as neutral element.

Kleene Algebra

Definition

An algebra $\langle K, +, \cdot, \star, 0, 1 \rangle$ of type $(2, 2, 1, 0, 0)$ is a Kleene algebra if for each $a, b, c \in K$ satisfy the following axioms:

(KA1) $a + (b + c) = (a + b) + c$;

(KA2) $a + b = b + a$;

(KA3) $a + a = a$;

(KA4) $a + 0 = 0 + a = a$;

(KA5) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$;

(KA6) $a \cdot 1 = 1 \cdot a = a$;

(KA7) $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$;

(KA8) $(a + b) \cdot c = (a \cdot c) + (b \cdot c)$;

(KA9) $a \cdot 0 = 0 \cdot a = 0$;

(KA10) $1 + (a \cdot a^\star) \leq a^\star$;

(KA11) $1 + (a^\star \cdot a) \leq a^\star$;

(KA12) If $a \cdot b \leq b$ then $a^\star \cdot b \leq b$;

(KA13) If $a \cdot b \leq a$ then $a \cdot b^\star \leq a$.

Where $\leq$ is the natural partial order on K defined by

$$a \leq b \text{ if and only if } a + b = b. \tag{3}$$

Elementary Properties of Kleene Algebra

Lemma

Let $\langle K, +, \cdot, \star, 0, 1\rangle$ be a Kleene algebra. Then

(KO1) *If $a \leq b$ then $a^\star \leq b^\star$.*

(KO2) $0^\star = 1$.

(KO3) $1 + a \cdot a^\star = 1 + a^\star \cdot a = a^\star$.

(KO4) $(a^\star)^\star = a^\star$.

(KO5) *If $a \cdot x = x \cdot b$ then $a^\star \cdot x = x \cdot b^\star$.*

(KO6) $(a + b)^\star = a^\star (b \cdot a^\star)^\star$.

Kleene operator based on Pseudo Uninorms

Definition

Let $\langle P, \leq \rangle$ be a join-semilattice, $e \in P$ and $U \in \mathfrak{U}_P^e$. A Kleene operator based on U is a function $\star : P \to P$ such that for each $x, y \in P$:

(K1) $e \vee U(x, x^\star) \leq x^\star$,

(K2) $e \vee U(x^\star, x) \leq x^\star$,

(K3) If $U(x, y) \leq y$ then $U(x^\star, y) \leq y$, and

(K4) If $U(y, x) \leq y$ then $U(y, x^\star) \leq y$.

Kleene operator based on Pseudo Uninorms

Proposition

Let $\langle P, \leq \rangle$ be a join-semilattice, $e \in P$ and $U \in \mathfrak{U}_P^e$ such that either U is a join morphism or $e \in \Delta_P$. If $U(x, x) \leq x$ for each $x \in P^e$ then the operator $x^\star = x \vee e$ is a Kleene operator for U.

Corollary

Let $\langle P, \leq \rangle$ be a join-semilattice, $e \in P$ and $U_1, U_2 \in \mathfrak{U}_P^e$ such that $e \in \Delta_P$. If $U_1(x, x) \leq x$ and $U_2(x, x) \leq x$ for each $x \in P^e$ then the operator $x^\star = x \vee e$ is a Kleene operator for both, $U_1 \rtimes U_2$ and $U_1 \ltimes U_2$.

Kleene Algebras based on Pseudo Uninorms

Theorem

Let $\langle P, \leq \rangle$ *be a join-semilattice,* $e \in P$ *and* $U \in \mathfrak{U}_P^e$ *such that either* U *is a join morphism or* $e \in \Delta_P$*. Then the operator* $x^\star = x \vee e$ *is a Kleene operator for* U *iff for each* $x, y \in P^e$*,* $U(x, y) = x \vee y$

Theorem

Let $\langle P, \leq, \bot, \top \rangle$ *be a bounded join-semilattice,* $e \in P$*,* $U \in \mathfrak{U}_P^e$ *be a join morphism such that* $U(\bot, \top) = U(\top, \bot) = \bot$ *and* $U(x, x) \leq x$ *for each* $x \in P^e$*. Then* $\langle P, \vee, U, \star, \bot, e \rangle$ *where* $x^\star = x \vee e$*, is a Kleene algebra.*

Theorem

Let $\langle K, +, U, \star, 0, e \rangle$ *be a Kleene algebra. Then*

- $U \in \mathfrak{U}_P^e$*;*
- $e \leq x^\star$*, for each* $x \in K$*;*
- $x^\star \geq e + x$*, for each* $x \in K$*;*
- $\star$ *is a closure operator on* $\langle K, \leq \rangle$*.*

Automorphisms

Definition

A function $\phi : P \to P$ is an automorphism on a poset $\langle P, \leq \rangle$ if it is bijective and for each $x, y \in P$ we have that

$$\phi(x) \leq \phi(y) \text{ if and only if } x \leq y. \tag{4}$$

- We will denote the set of all automorphism on $\langle P, \leq \rangle$ by $Aut\langle P, \leq \rangle$.
- The inverse of an automorphism is also an automorphism.

  The composition of two automorphism is also an automorphism.

  The identity function Id_P on P is an automorphism. In addition, $\phi \circ Id_P = \phi = Id_P \circ \phi$.
- Therefore, $\langle Aut\langle P, \leq \rangle, \circ \rangle$ is a group.

Automorphisms acting on pseudo uninorms

- The **action of ϕ on a function** $f : P^n \to P$, denoted by f^ϕ, is the function:

$$f^\phi(x_1, \ldots, x_n) = \phi^{-1}(f(\phi(x_1), \ldots, \phi(x_n))). \quad (5)$$

Proposition

Let U be a pseudo uninorm and ϕ be an automorphism on a bounded poset $\langle P, \leq \rangle$. Then U^ϕ is also a pseudo uninorm. In addition,

- *if $\langle P, \leq \rangle$ is bounded then $\phi(\bot) = \bot$ and $\phi(\top) = \top$.*
- *if $\langle P, \leq \rangle$ is a join semilattice then $\phi(x \vee y) = \phi(x) \vee \phi(y)$, i.e. ϕ is a join morphism.*
- *if $\langle P, \leq \rangle$ is a meet semilattice then $\phi(x \wedge y) = \phi(x) \wedge \phi(y)$, i.e. ϕ is a meet morphism.*

Automorphisms acting on Kleene algebras

Proposition

Let U be a pseudo uninorm and ϕ be an automorphism on a bounded poset $\langle P, \leq \rangle$. Then U^{ϕ} is also a pseudo uninorm. In addition,

- *if $\langle P, \leq \rangle$ is bounded then $\phi(\bot) = \bot$ and $\phi(\top) = \top$.*
- *if $\langle P, \leq \rangle$ is a join semilattice then $\phi(x \vee y) = \phi(x) \vee \phi(y)$, i.e. ϕ is a join morphism.*
- *if $\langle P, \leq \rangle$ is a meet semilattice then $\phi(x \wedge y) = \phi(x) \wedge \phi(y)$, i.e. ϕ is a meet morphism.*

Theorem

Let $\langle P, \leq \rangle$ be a join-semilattice, $e \in P$, $U \in \mathfrak{U}_P^e$ and $\phi \in Aut\langle P, \leq \rangle$. If $\star : P \to P$ is a Kleene operator for U then $\circledast : P \to P$, defined by $x^{\circledast} = \phi^{-1}(\phi(x)^{\star})$ is a Kleene operator based on U^{ϕ}.

Final Remarks

- We shown the relation between the notions of Kleene algebras and pseudo uninorms.
- We shown that every Kleene algebra induces a pseudo uninorm and that *some* pseudo uninorms induce Kleene algebras.
- This connection enables both:

  (1) Another viewpoint on the theory of Kleene algebras, and
  (2) A way to build Kleene algebras in the fuzzy setting

  Future works:

  (1) New ways of generate Kleene algebras
  (2) New operators on Kleene algebras.
  (3) It is possible to define Kleene algebras over the set of all Kleene algebras over a fixed poset?


THANKS!